A new and simple proof of Schauder's theorem

Volker Runde

Schauder's theorem from 1930 ([11]) asserts that a bounded linear operator between Banach spaces is compact if and only if its adjoint is. Schauder's original proof is completely elementary; at its heart is a diagonal argument that is reminiscent of proofs of the Arzelà–Ascoli theorem. Indeed, in [4], S. Kakutani gave a proof of Schauder's theorem that invokes the Arzelà–Ascoli theorem explicitly. This proof has by now become the canonical one, i.e., it is featured in most textbooks on functional analysis (see [9], [7], or [2], for instance). An alternative proof of Schauder's theorem uses the Alaoğlu–Bourbaki theorem ([1]).

In this note, we shall provide another proof of Schauder's theorem, which is both short and completely elementary in the sense that it does not depend on anything beyond basic functional analysis, i.e., the Hahn–Banach theorem and some of its consequences; in particular, we avoid the Arzelà–Ascoli theorem (and any kind of related diagonal argument).

Throughout, we write $\mathrm{Ball}(E)$ for the closed unit ball of a Banach space E.

The following observation was made by H. Saar ([10]).

Let E and F be Banach spaces, and let $T\colon E \to F$ be compact. Then $T(\mathrm{Ball}(E))$ is totally bounded, so that, for each $\epsilon > 0$, there are $x_1, \ldots, x_n \in \mathrm{Ball}(E)$ such that, for each $x \in \mathrm{Ball}(E)$, there is $j \in \{1, \ldots, n\}$ such that $\|Tx - Tx_j\| < \epsilon$. Letting $Y_\epsilon := \mathrm{span}\{Tx_1, \ldots, Tx_n\}$. It follows that $\|Q_{Y_\epsilon} T\| < \epsilon$, where $Q_{Y_\epsilon}\colon F \to F/Y_\epsilon$ is the quotient map.

Conversely, suppose that $T\colon E \to F$ is bounded and that, for each $\epsilon > 0$, there is a finite-dimensional subspace Y_ϵ of F such that $\|Q_{Y_\epsilon} T\| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then there is a finite-dimensional subspace $Y_{\frac{\epsilon}{3}}$ such that $\|Q_{Y_{\frac{\epsilon}{3}}} T\| < \frac{\epsilon}{3}$. Set

$$K := \left\{ y \in Y_{\frac{\epsilon}{3}} : \mathrm{dist}(y, T(\mathrm{Ball}(E))) < \frac{\epsilon}{3} \right\}.$$

Then K is bounded and thus, as $\dim Y_{\frac{\epsilon}{3}} < \infty$, totally bounded, i.e., there are $y_1, \ldots, y_m \in K$ such that, for each $y \in K$, there is $j \in \{1, \ldots, m\}$ such that $\|y - y_j\| < \frac{\epsilon}{3}$. By the definition of K, there is, for each $j = 1, \ldots, m$, an element $x_j \in \mathrm{Ball}(E)$ with $\|y_j - Tx_j\| < \frac{\epsilon}{3}$. Let $x \in \mathrm{Ball}(E)$ be arbitrary. Since $\|Q_{Y_{\frac{\epsilon}{3}}} T\| < \frac{\epsilon}{3}$, there is $y \in K$ such that $\|Tx - y\| < \frac{\epsilon}{3}$. Let $j \in \{1, \ldots, m\}$ such that $\|y - y_j\| < \frac{\epsilon}{3}$. It follows that

$$\|Tx - Tx_j\| \leq \|Tx - y\| + \|y - y_j\| + \|y_j - Tx_j\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Hence, $T(\mathrm{Ball}(E))$ is totally bounded, and T is compact.

We thus have:

Lemma 1. *Let E and F be Banach spaces, and let $T\colon E \to F$ be bounded. Then T is compact if and only if, for each $\epsilon > 0$, there is a finite-dimensional subspace Y_ϵ of F such that $\|Q_{Y_\epsilon} T\| < \epsilon$, where $Q_{Y_\epsilon}\colon F \to F/Y_\epsilon$ is the quotient map.*

Our next lemma requires the Hahn–Banach theorem.

Lemma 2. *Let E and F be Banach spaces, let $T\colon E \to F$ be bounded, let $\epsilon > 0$, and let X be a closed subspace of E with finite codimension such that $\|T|_X\| < \frac{\epsilon}{3}$. Then there is a finite-dimensional subspace X_0 of E such that $\|Q_{TX_0} T\| < \epsilon$, where $Q_{TX_0}\colon F \to F/TX_0$ is the quotient map.*

Proof. Using the Hahn–Banach theorem, we can embed F isometrically into $\ell^\infty(\mathbb{I})$ for a suitable index set ($\mathbb{I} = \mathrm{Ball}(E^*)$ will do). Hence, we can suppose without loss of generality that $F = \ell^\infty(\mathbb{I})$.

Applying the Hahn–Banach theorem coordinatewise, we obtain an operator $\tilde{T}\colon E \to \ell^\infty(\mathbb{I})$ such that $\tilde{T}|_X = T|_X$ and $\|\tilde{T}\| = \|T|_X\| < \frac{\epsilon}{3}$. Set $S := T - \tilde{T}$. Then S vanishes on X, and since X has finite codimension this means that S is a finite rank operator and thus compact. Consequently, there are $x_1, \ldots, x_n \in \mathrm{Ball}(E)$ such that, for each $x \in \mathrm{Ball}(E)$, there is $j \in \{1, \ldots, n\}$ with $\|Sx - Sx_j\| < \frac{\epsilon}{3}$.

Fix $x \in \mathrm{Ball}(E)$, let $j \in \{1, \ldots, n\}$ be such that $\|Sx - Sx_j\| < \frac{\epsilon}{3}$, and note that

$$\|Tx - Tx_j\| \leq \|Sx - Sx_j\| + \|\tilde{T}x - \tilde{T}x_j\| < \frac{\epsilon}{3} + 2\|\tilde{T}\| < \frac{\epsilon}{3} + 2\frac{\epsilon}{3} = \epsilon.$$

The space $X_0 = \mathrm{span}\{x_1, \ldots, x_n\}$ thus has the desired property. □

In conjunction, Lemmas 1 and 2 yield immediately:

Corollary. *Let E and F be Banach spaces, and let $T\colon E \to F$ be bounded with the following property: for each $\epsilon > 0$, there is a closed subspace X_ϵ of E with finite codimension such that $\|T|_{X_\epsilon}\| < \epsilon$. Then T is compact.*

We can now prove Schauder's theorem:

Theorem. *Let E and F be Banach spaces, and let $T : E \to F$ be a bounded linear operator. Then the following are equivalent:*

(i) *T is compact;*

(ii) *for each $\epsilon > 0$, there is a finite-dimensional subspace Y_ϵ of F such that $\|Q_{Y_\epsilon} T\| < \epsilon$, where $Q_{Y_\epsilon}\colon F \to F/Y_\epsilon$ is the quotient map;*

(iii) *for each* $\epsilon > 0$*, there is a closed subspace* X_ϵ *of* E *with finite codimension such that* $\|T|_{X_\epsilon}\| < \epsilon$*;*

(iv) $T^* \colon F^* \to E^*$ *is compact.*

Proof. (i) $\Longleftrightarrow$ (ii) is Lemma 1, and (iii) $\Longrightarrow$ (ii) follows from Lemma 2.

(ii) $\Longrightarrow$ (iv): Let $\epsilon > 0$, and let Y_ϵ be a finite-dimensional subspace of F such that $\|Q_{Y_\epsilon} T\| < \epsilon$. Let X_ϵ be the annihilator of Y_ϵ in F^*, so that X_ϵ has finite codimension in F^*, $T^*|_{X_\epsilon} = (Q_{Y_\epsilon} T)^*$, and thus $\|T^*|_{X_\epsilon}\| < \epsilon$. Since $\epsilon > 0$ was arbitrary, the Corollary—applied to T^*—thus yields (iv).

(iv) $\Longrightarrow$ (iii): Let $\epsilon > 0$. Invoking Lemma 1 for T^* and then arguing as in the proof of (ii) $\Longrightarrow$ (iv), we obtain a closed subspace X_ϵ of E^{**} with finite codimension such that $\|T^{**}|_{X_\epsilon}\| < \epsilon$. Consequently, $\|T|_{X_\epsilon \cap E}\| < \epsilon$ holds as well. Since $\epsilon > 0$ was arbitrary, the Corollary implies (i). $\square$

Remarks. 1. As M. Cwikel pointed out to me, the equivalence of (i) and (iii) in the Theorem was already obtained by H. E. Lacey in [6]. His result is reproduced with a proof on [8, p. 91].

2. A similar proof of Schauder's Theorem—in the sense that it is both elementary and avoids diagonal arguments—is given in [3] and [5]: this was pointed out to me by D. Werner and A. Valette, respectively.

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Author's address: Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

E-mail: `vrunde@ualberta.ca`

URL: `http://www.math.ualberta.ca/~runde/`